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Technical Notes

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The Inverse Problem for Supersonic Airfoils

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Introduction

THE inverse or design problem for the case of two-dimensional supersonic airfoil shapes is considered. In view of the hyperbolic structure of the underlying equations, the calculation is simpler than the corresponding subsonic problem. In a recent paper,[1] the authors developed a numerical procedure for treating the direct problem, the calculation of supersonic flowfields past given profiles. This procedure makes use of streamlines as one of the coordinates. As a result, it is especially suited to the inverse problem. Since the adaptation of the method to the present problem is very similar to the original formulation, we will give only a brief outline of the procedure in this Note. For purposes of comparison, we also present two approximate solutions of the problem, one a simple treatment based on linearized analysis and the other based on shock expansion theory. The latter proves to be highly accurate and fails only in extreme cases.

Outline of the Method

Consider two-dimensional supersonic flow, which we describe by the flow deflection angle θ, the Mach angle $\mu = \sin^{-1}(1/M)$, and s, the entropy divided by the gas constant R.

The equations of motion in characteristic form are[2]

$$\mathrm{d}r^{\pm} = \pm \frac{\sin 2\mu}{2\gamma}\,\mathrm{d}s \text{ on } C^{\pm}: \ \frac{\mathrm{d}y}{\mathrm{d}x} = \tan(\theta \pm \mu) \tag{1}$$

$$\mathrm{d}s = 0 \text{ on streamlines: } \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta \tag{2}$$

and where $r^{\pm} = \theta \pm P(\mu)$ and $P(\mu)$, the Prandtl angle, is defined by

$$P(\mu) = \lambda^{1/2} \tan^{-1}(\lambda^{1/2} \tan\mu) - \mu, \qquad \lambda = (\gamma+1)/(\gamma-1) \tag{3}$$

We introduce a coordinate system made up of the streamlines (α = const) and the C^+ characteristics (β = const). The equations then become

$$s_\beta = 0 \tag{4}$$

$$r^+_\alpha = \frac{\sin 2\mu}{2\gamma} s_\alpha \tag{5}$$

$$r^-_\alpha + w r^-_\beta = -\frac{\sin 2\mu}{2\gamma} s_\alpha, \quad w = -\frac{2}{1-\tan\theta\tan\mu}\frac{x_\alpha}{x_\beta} \tag{6}$$

In this formulation, the coordinates (x,y) in the physical plane become dependent variables and are governed by the equations

$$y_\beta = x_\beta \tan\theta, \qquad y_\alpha = x_\alpha \tan(\theta+\mu) \tag{7}$$

Equations (4-7) are to be augmented by the shock relations, which are not repeated here, and by the given airfoil pressure distribution $p = p_0(x)$.

The transformation from the physical to the (α,β) plane leaves open two arbitrary functions. One of these is fixed by setting

$$x(0,\beta) = \beta \tag{8}$$

on the streamline $\alpha = 0$, which we take to be the as yet unknown airfoil. As a second condition, we require that the shock angle η vary linearly with α,

$$\eta(\alpha) = \eta(0) + [\mu_0 - \eta(0)]\alpha, \quad 0 \le \alpha \le 1 \tag{9}$$

The method of solution is iterative and begins with an approximate solution. This approximation is allied to shock expansion theory.[3,4] Figure 1 contains a sketch of the plane of integration. The curve $\alpha(\beta)$ represents the as yet unknown shock trajectory. As indicated in Fig. 1, a uniform α mesh is chosen, which with $\alpha(\beta)$ generates the β mesh over the first portion of the figure. The determination of the solution starts with the replacement of Eq. (6) by the approximation

$$r^- = \theta - P(\mu) = -P_0(\alpha) \tag{10}$$

where $-P_0(\alpha)$ is the value determined at the shock. If Eq. (10) is substituted into Eq. (5), we obtain

$$2\frac{\partial}{\partial\alpha} P(\mu) = P_0'(\alpha) + \frac{\sin 2\mu}{2\gamma} s'(\alpha) \tag{11}$$

where $s(\alpha)$ is also determined from the shock relations.

For each given value of β, this is an ordinary differential equation for μ, which can be solved numerically. Initial and final values $\mu[\alpha(\beta),\beta]$ and $\mu(0,\beta)$ are both known, which allows us to determine the shock $\alpha(\beta)$. The values $\mu[\alpha(\beta),\beta]$ come from the shock conditions and the values $\mu(0,\beta)$ from the following relation between μ, s, and the normalized pressure p:

$$\sin^2\mu = \frac{\gamma-1}{2} \frac{\exp\left[\frac{\gamma-1}{\gamma}(s+\ln p)\right]}{1+\frac{\gamma-1}{2}M_0^2 - \exp\left[\frac{\gamma-1}{\gamma}(s+\ln p)\right]} \tag{12}$$

Since $s(0)$ is known and $p(0,\beta) = p_0(\beta)$ is given, this determines $\mu(0,\beta)$.

On the portion of $\alpha = 0$ not lying under the front shock, we choose a uniform β mesh. We use Eq. (12) to determine $\mu(0,\beta)$ and integrate Eq. (11) up from $\alpha = 0$ to finish the determination of $\mu(\alpha,\beta)$. Equation (10) is used to determine $\theta(\alpha,\beta)$. The approximate solution is then known everywhere

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in the (α,β) plane. Finally, Eqs. (7) are integrated to find the transformation to the physical plane everywhere. In particular, the body shape $f(x)$ is given by

$$f(x)=f(0)+\int_0^x \tan\theta(0,\beta)\,\mathrm{d}\beta \tag{13}$$

which completes the computation of the approximate solution.

The iteration procedure starts with the neglected Eqs. (6), which were replaced in the approximate solution by Eq. (10). We numerically integrate Eqs. (6) downstream along the C^- characteristics, starting at the shock. This produces new values of $r^-(\alpha,\beta)$ everywhere. From $r^-(0,\beta)$ we can determine

$$r^+(0,\beta)=r^-(0,\beta)+2P[\mu(0,\beta)] \tag{14}$$

since $\mu(0,\beta)$ is given by Eq. (12). Equation (5) is then integrated up from the body along the C^+ characteristics to give $r^+(\alpha,\beta)$ everywhere. We can then obtain improved values of θ and μ everywhere from

$$\theta=\tfrac{1}{2}(r^+-r^-),\qquad \mu=P^{-1}[\tfrac{1}{2}(r^+-r^-)] \tag{15}$$

A new shock angle $\eta(\alpha)$ is determined from the shock relations, and a new transformation to the physical plane is obtained from Eqs. (13) and (7). The iteration is then repeated until a convergence criterion is met.

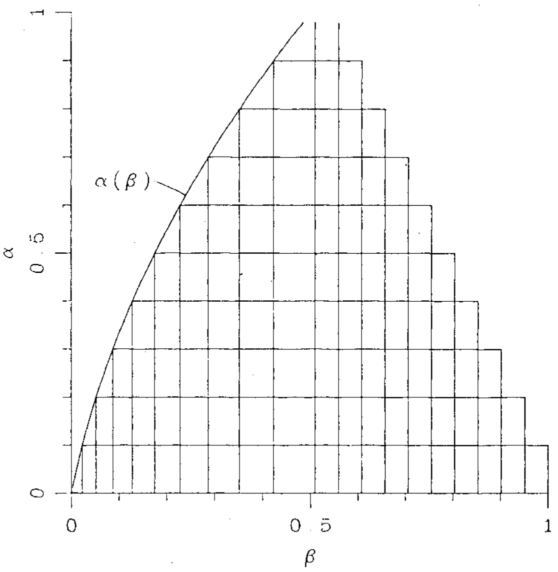

Fig. 1 Plane of integration: lines $\alpha=$const represent streamlines, lines $\beta=$const represent C^+ characteristics, and curve $\alpha(\beta)$ represents the bow shock.

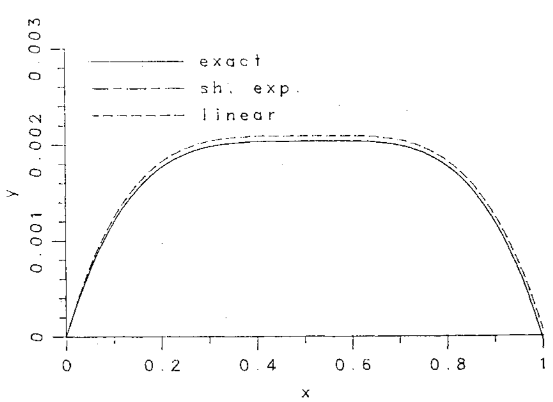

Fig. 2 Airfoil shapes computed for $M_0=1.2$ and airfoil pressure distribution $\ln p_0(x)=0.05(1-2x)^3$.

Approximate Calculations

Linearized Approximation

Perhaps the simplest calculation for $f(x)$ given the pressure distribution $p_0(x)$ follows from linearized theory.[4] In our normalization, linearized theory

$$f'(x)=\frac{\sqrt{M_0^2-1}}{\gamma M_0^2}[p_0(x)-1]$$

or

$$f(x)=f(0)+\frac{\sqrt{M_0^2-1}}{\gamma M_0^2}\left(\int_0^x p_0(\xi)\,\mathrm{d}\xi-x\right) \tag{16}$$

Shock Expansion Approximation

One may also base an approximate calculation of $f(x)$ on shock expansion theory.[4] We recall that if $\theta_0(x)$ and $\mu_0(x)$ denote values on the airfoil, then in this approximation it is assumed that

$$\theta_0(x)-P[\mu_0(x)]=-P_0 \tag{17}$$

where P_0 is the value at the leading edge behind the shock. Equation (12) determines $\mu_0(x)$ from the given pressure $p_0(x)$ and the entropy at the leading edge s_0, and then $f(x)$ follows from

$$f(x)=f(0)+\int_0^x \tan[P(\mu_0(\xi))-P_0]\,\mathrm{d}\xi \tag{18}$$

As an examination shows, $f(x)$ calculated in this way agrees with the first approximation in the iterative solution outlined in the previous section.

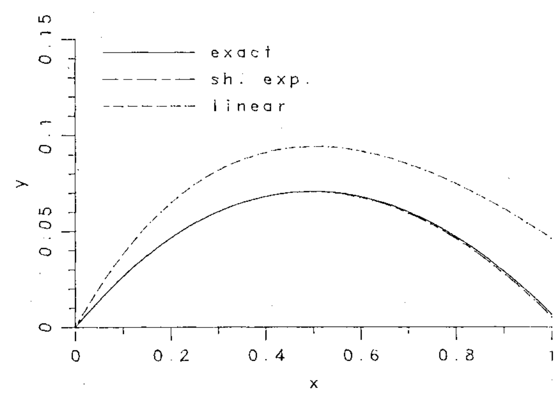

Fig. 3 Airfoil shapes computed for $M_0=2.5$ and airfoil pressure distribution $\ln p_0(x)=1-2x$.

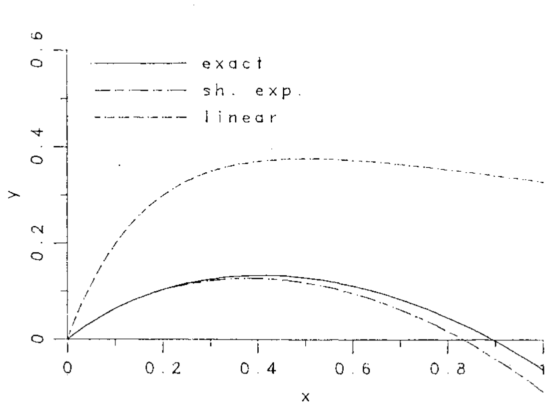

Fig. 4 Airfoil shapes computed for $M_0=5$ and airfoil pressure distribution $\ln p_0(x)=3(1-2x)$.

Results

Sample calculations are shown in Figs. 2-4. In each the results obtained from linearized theory, shock expansion theory, and the exact numerical calculation are compared.

Figure 2 contains the results for a low Mach number ($M_0 = 1.2$) and small pressure jump. As should be expected, all of the results are in close agreement. For Fig. 3, the Mach number is moderate ($M_0 = 2.5$) and the jump in $\ln p$ at the leading edge is unity. In this case, linear theory is poor in predicting an overly thick body. The result based on shock expansion theory, on the other hand, is virtually indistinguishable from the exact case. In the final example (Fig. 4), both the upstream Mach number ($M_0 = 5$) and the pressure jump are relatively large. Linearized theory is now very poor. Shock expansion theory still does quite well for most of the derived airfoil and begins to depart significantly only near the trailing edge.

Conclusions

A method for the design of two-dimensional supersonic airfoils has been presented, which incorporates available physical and mathematical knowledge of the problem (e.g., shock expansion theory and characteristics), in order to facilitate the numerical computation. A similar approach should prove useful in the more complicated problem of the design of real airfoils in which three-dimensional flow, boundary-layer effects, etc., must be considered. The iterative nature of the present method, in particular, makes it well suited to the inclusion of boundary-layer corrections.

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Sidewall Effects on Airfoil Tests

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Introduction

AIRFOIL tests in a wind tunnel may be affected by two fundamentally different types of wall interference: 1) effects from top and bottom walls and 2) sidewall effects. Interference effects from top and bottom walls may be treated as an entirely two-dimensional and virtually nonviscous problem. New theoretical methods, as well as the adaptive wall technique, are fairly promising in providing a solution to this problem.[1]

Sidewall effects, on the other hand, have to be attributed to viscous effects and introduce three-dimensional disturbances into the flow. One way of treating this problem is the use of sidewall boundary-layer control, such as distributed suction in the area of the model.[2,3] A disadvantage of this solution is that schlieren visualizations and laser velocimetry are precluded. Therefore, theoretical methods which will allow us to estimate the interference effects and to correct the experimental results are very desirable. A recent review of the methods available is given in Refs. 4 and 5.

One assumption in all of these theories is that sidewall interference effects may be accounted for by some global correction to the mainstream flow condition. Usually a correction to incidence (or lift or normal force) is given and, in addition, some theories estimate a Mach number correction.

The intention of this Note is to contribute to an evaluation of the theoretical methods by presenting experimental results that allow a judgment to be made on the basic assumptions used for the theories.

Wind Tunnel and Test Setup

Most of the experiments were made in the DFVLR 1×1 m Transonic Wind Tunnel in Göttingen with a solid wall configuration.[6] The CAST 7 airfoil model used had a chord of 15 cm. The span of the model was 1 m, but side plates have been inserted to vary the effective span. The plates extended from the top to the bottom wall, as well as 0.5 m upstream and downstream of the model, giving a boundary-layer displacement thickness of $\delta^* = 1$ mm at the model. The plates were set at distances of $b = 15$, 30, and 60 cm, which resulted in model aspect ratios of $\Lambda = 1,2$, and 4, and $\delta^*/b = 0.013$, 0.007, and 0.003. A few complementary tests were made in the TU-Berlin Transonic Wind Tunnel. This tunnel has a test section of 0.15×0.15 m with adaptive flexible top and bottom walls.[1] The model used was a 10 cm chord CAST 7 airfoil, giving an aspect ratio of $\Lambda = 1.5$. The Reynolds number for all tests presented here was $Re = 1.2\text{-}1.4 \times 10^6$. Transition was fixed at

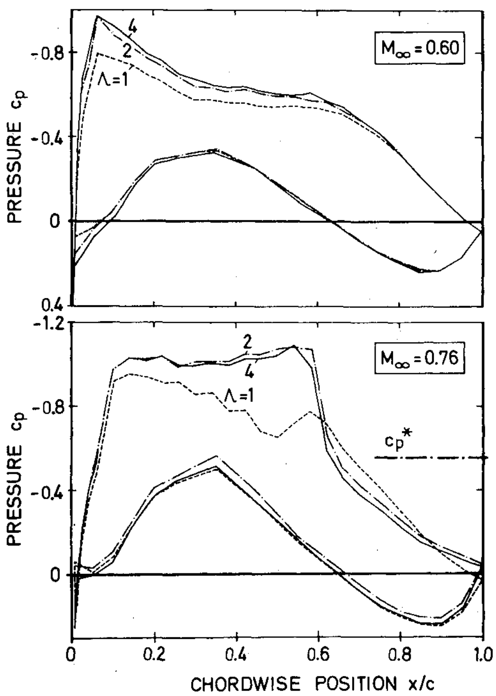

Fig. 1 Surface pressure distributions for CAST 7 airfoil; $\alpha = 1$ deg, various aspect ratios.

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